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Stability and the negative mode for Schwarzschild in a finite cavity

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Abstract

It has been proposed that translationally-invariant black branes are classically stable if and only if they are locally thermodynamically stable. Reall has outlined a general argument to demonstrate this, and studied in detail the case of charged black p-branes in type II supergravity [?]. We consider the application of his argument in the simplest non-trivial case, an uncharged asymptotically flat brane enclosed in a finite cylindrical cavity. In this simple context, it is possible to give a more complete argument than in the cases considered earlier, and it is therefore a particularly attractive example of the general approach.

1 Introduction

It was recently proposed, in a conjecture of Gubser and Mitra [?, ?], that a black brane with a non-compact translational symmetry is classically stable if, and only if, it is locally thermodynamically stable. In subsequent work [?], Reall provided a general argument for the validity of this conjecture, based on establishing a relationship between the classical Gregory-Laflamme instability [?, ?] and a Euclidean negative mode associated with thermodynamic instability. Interesting tests of this conjecture are provided by solutions which pass from (classical or thermodynamic) stability to instability as some parameter is varied. In [?], charged black p-brane solutions were considered, where in some cases the specific heat becomes positive near extremality, and it was explicitly demonstrated that for suitable gauge choices, the equations satisfied by the Euclidean negative mode and the classical Gregory-Laflamme perturbation are equivalent.

In this paper, we will consider the simpler case of an uncharged black p-brane in $d+p$ dimensions, constructed by taking a direct product of a d-dimensional Schwarzschild black hole and p flat directions. We consider this system in a finite cavity, so we can

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vary the specific heat by varying the ratio of the cavity size and the black hole's mass. The relation between the classically unstable mode and the Euclidean negative mode for this solution was already considered in infinite volume in [?]. We show that the introduction of the cavity walls does not spoil the equivalence, and furthermore show that the negative mode coincides precisely with the regime of thermodynamic instability.[1]

To begin, we briefly review the discussion in [?]. In the semi-classical approximation to the Euclidean path integral

$$Z = \int D[g] e^{-I[g]}, \tag{1}$$

we write the metric as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is some Euclidean solution of the field equations and $h_{\mu\nu}$ is a small perturbation. The action can then be approximated by

$$I_E[g] = I_E[\bar{g}] + \int d^4x \sqrt{\bar{g}} h^{\mu\nu} A_{\mu\nu\rho\sigma} h^{\rho\sigma}. \tag{2}$$

If the operator $A_{\mu\nu\rho\sigma}$ in the quadratic term has negative eigenvalues, there will be an imaginary part in the partition function, and the classical solution $\bar{g}_{\mu\nu}$ is interpreted as a saddle-point. If we decompose the perturbation into a transverse tracefree part $h^{TT}_{\mu\nu}$, a trace and a longitudinal part, the trace will have a negative quadratic term, but this is just a sign of the usual conformal factor problem, and the resulting imaginary part is cancelled by a corresponding contribution from integrating over ghosts, so it does not correspond to a physical instability [?]. The quadratic term for the longitudinal part is positive definite. The physical negative modes in pure gravity therefore arise only from the quadratic term for $h^{TT}_{\mu\nu}$, which involves the Euclidean Lichnerowicz operator

$$G_{\mu\nu\rho\sigma} = -\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}\nabla_\beta\nabla^\beta - 2\bar{R}_{\mu\rho\nu\sigma}. \tag{3}$$

The negative modes are given by the eigenvectors of

$$G_{\mu\nu}{}^{\rho\sigma} h^{TT}_{\rho\sigma} = \lambda h^{TT}_{\mu\nu} \tag{4}$$

with negative eigenvalues.

In [?], it was shown that Schwarzschild black holes with negative specific heat will necessarily have a negative mode. This was extended to the charged black brane solutions in [?]. Thus, thermodynamic instability in the canonical ensemble implies the existence of a negative mode. However, no general proof of the converse exists. In this paper we will check this explicitly by finding the lowest eigenvalue of (4) and showing it is negative if and only if the specific heat is negative.

The key remaining step in relating thermodynamic instability and classical instability is relating the negative mode satisfying (4) in the black hole solution to the

[1] A similar calculation of the negative mode for Euclidean Schwarzschild-anti de Sitter was carried out in [?]. It was also shown in [?] that for Reissner-Nordström black holes in infinite volume, the negative mode coincides with the regime of thermodynamic instability.

unstable mode in the Gregory-Laflamme analysis of the perturbations about a black brane solution. For the uncharged black branes we are interested in, this connection is very direct [?]. The metric for the brane is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \delta_{ij} dz^i dz^j, \tag{5}$$

where $g_{\mu\nu}$ is the d-dimensional Lorentzian Schwarzschild metric and z^i are p flat spatial directions. If we consider a metric perturbation

$$h_{\mu\nu} = \exp\left(i\mu_i z^i\right) H_{\mu\nu}, \qquad h_{\mu i} = h_{ij} = 0, \tag{6}$$

where $H_{\mu\nu}$ is transverse traceless and spherically symmetric, then the equation of motion implies [?, ?]

$$\tilde{G}_{\mu\nu}{}^{\rho\sigma} H_{\rho\sigma} = -\mu^2 H_{\mu\nu}, \tag{7}$$

where $\tilde{G}$ is now the Lorentzian Lichnerowicz operator for the metric $g_{\mu\nu}$. A time-independent solution of this equation with non-zero μ represents a threshold unstable mode, separating stable and unstable perturbations. But on time-independent perturbations, (4) and (7) are equivalent. Thus the conditions for classical instability and existence of a Euclidean negative mode give rise to the same equation.

The introduction of a finite boundary introduces little modification in this argument connecting the Euclidean negative mode in the black hole solution and the classical Gregory-Laflamme instability of the uncharged black brane. For the case of a black hole in a finite cavity, the appropriate boundary condition in both the Euclidean negative mode and the classical perturbation is that the induced metric on the boundary be unchanged. In the remainder of the paper, we will show that this Euclidean negative mode (and hence the classical instability) occurs precisely when the black brane is thermodynamically unstable.

2 Schwarzschild in a finite cavity

Gravitational thermodynamics in a finite cavity was considered by York [?]. He considered four-dimensional spacetime, with the proper area of the spherical cavity $A = 4\pi r_b^2$ and the local temperature T at the cavity wall fixed. In the Euclidean solution, these boundary conditions amount to fixing the induced metric on the boundary. If the product of cavity radius and temperature was sufficiently high, $r_b T > \sqrt{27}/8\pi$, there were two Schwarzschild solutions of mass M_1, M_2 which satisfied the boundary conditions. As $r_b T$ runs from this minimum value to infinity, M_1/r_b runs from 1/3 to 0, and M_2/r_b runs from 1/3 to 1/2 (thus, in the limit, the black hole fills the cavity). The stability in the canonical ensemble was analysed by calculating the specific heat at constant area of the boundary, with the result

$$C_A = 8\pi M^2 \left(1 - \frac{2M}{r_b}\right)\left(\frac{3M}{r_b} - 1\right)^{-1}. \tag{8}$$

Thus, the smaller black hole of mass M_1 has negative specific heat and is thermodynamically unstable, while the larger black hole of mass M_2 has positive specific heat and is thermodynamically stable.

In considering perturbations, it is more convenient to use the black hole's mass M and r_b as parameters, rather than T and r_b, and we will imagine varying r_b. We can consider any $r_b > 2M$. For $2M < r_b < 3M$, we have a positive specific heat, while for $r_b > 3M$, we have a negative specific heat. We therefore expect to see a negative mode only for $r_b > 3M$. Allen analysed the problem of finding the negative mode for black holes in a finite cavity prior to York's study of the thermodynamics [**?**]. He found that the negative mode exists if $r_b \gtrapprox 2.89M$. A resolution of this apparent contradiction was presented by York [**?**], who observed that Allen had fixed the coordinate location of the boundary, which, in his choice of gauge for the perturbation, did not correspond to fixing the area of the boundary.

In the next section, we will discuss the solution of the eigenvalue equation (4) with the induced metric on the boundary fixed. First, however, we will briefly discuss the straightforward extension of York's thermodynamic analysis to higher dimensions. The d-dimensional Schwarzschild black hole is

$$ds^2 = \left(1 - \frac{\omega M}{r^{d-3}}\right) dt^2 + \left(1 - \frac{\omega M}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}, \tag{9}$$

where $\omega = 16\pi/[(d-2)V_{d-2}]$ and V_{d-2} is the volume of the unit S^{d-2}. Hence, the relation between mass and temperature is

$$T(r_b) = \frac{1}{4\pi(\omega M)^{1/(d-3)}} \left(1 - \frac{\omega M}{r_b^{d-3}}\right)^{-1/2}, \tag{10}$$

and the specific heat of a black hole in a cavity of radius r_b is

$$C_A = 4\pi(d-3)M(\omega M)^{1/(d-3)} \left(1 - \frac{\omega M}{r_b^{d-3}}\right) \left(\frac{d-1}{2}\frac{\omega M}{r_b^{d-3}} - 1\right)^{-1}. \tag{11}$$

We therefore expect there to be a negative mode in the Euclidean solution only for $r_b^{d-3} > \frac{d-1}{2}\omega M$.

3 Finding the Negative Mode

We will now discuss the calculation of the negative mode for a d-dimensional Euclidean Schwarzschild black hole in a finite cavity with the induced metric on the wall of the cavity fixed. By the foregoing discussion, this also gives a threshold unstable mode for the uncharged black brane solution. We will first discuss the analysis for four dimensions in detail and then sketch the extension to higher dimensions.

3.1 Four dimensions

We wish to consider perturbations of the four-dimensional Euclidean Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{12}$$

We are looking for perturbations which are eigenvectors of (4) with negative eigenvalues. This problem was analysed in [?] for the black hole in infinite volume, and it was found that in an expansion in spherical harmonics, the only part which can have a negative eigenvalue is the spherically symmetric static mode. In considering the black hole in a finite box, it will therefore suffice for us to consider a perturbation

$$h^{\mu}{}_{\nu} = Diag\left[H_0(r), H_1(r), -\frac{1}{2}\left(H_0(r) + H_1(r)\right), -\frac{1}{2}\left(H_0(r) + H_1(r)\right)\right], \tag{13}$$

where the condition of transversality, $\nabla_\mu h^{\mu}{}_{\nu} = 0$, implies

$$H_0(r) = \left[-\frac{r(r-2M)}{r-3M}\frac{d}{dr} - \frac{3r-5M}{r-3M}\right] H_1(r). \tag{14}$$

The task of finding the negative mode then reduces to finding negative eigenvalues of

$$\left[-\left(1 - \frac{2M}{r}\right)\frac{d^2}{dr^2} - \frac{2(r-4M)(2r-3M)}{r^2(r-3M)}\frac{d}{dr} + \frac{8M}{r^2(r-3M)}\right] H_1(r) = \lambda H_1(r). \tag{15}$$

This equation has regular singular points at $r = 0$, $r = 2M$ & $r = 3M$ and an irregular singular point at $r = \infty$, so an explicit solution to it can not be found. As in previous works, we must therefore use numerical methods to find the eigenvalues. In the infinite cavity the required boundary conditions for the solution are regularity at the horizon, $r = 2M$, and normalizability at infinity. Imposing these conditions on the solution then yields one negative eigenvalue, $\lambda \approx -0.19M^{-2}$ [?].

To search for negative modes in a finite cavity, we need to impose the condition that the induced metric on the walls of the cavity is fixed. This is complicated by the fact that the perturbation is generically non-zero in the spherical directions. In the unperturbed metric the area of a spherical cavity of radius $r = r_b$ is $A = 4\pi {r_b}^2$. The perturbed metric is

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \epsilon H_0(r)\right) d\tau^2 + \frac{1 + \epsilon H_1(r)}{1 - \frac{2M}{r}} dr^2 + r^2\left[1 - \frac{1}{2}\epsilon(H_0(r) + H_1(r))\right] d\Omega^2, \tag{16}$$

and so after the perturbation the surface $r = r_b$ has area

$$A = 4\pi {r_b}^2\left[1 - \frac{1}{2}\epsilon(H_0(r) + H_1(r))\right]. \tag{17}$$

The change of area could be corrected by allowing the boundary of the cavity r_b to move to r_b' after the perturbation, where r_b' is chosen such that the area of the cavity wall in the perturbed metric is the same as that in the unperturbed metric at r_b. We will instead give a more careful analysis by making a change of coordinates to work in a different gauge for the perturbation, in which the area of the cavity wall at fixed radius is unchanged by the perturbation. The appropriate coordinate transformation is

$$r \to \rho : \quad \rho^2 = r^2 \left[1 - \frac{1}{2}\epsilon (H_0(r) + H_1(r)) \right]. \tag{18}$$

Rewriting the perturbed metric in the new coordinates and keeping only the dominant first order terms of the metric perturbation gives

$$ds^2 = \left(1 - \frac{2M}{\rho}\right)(1 + \epsilon F_0(\rho))\, d\tau^2 + \left(1 - \frac{2M}{\rho}\right)^{-1} (1 + \epsilon F_1(\rho))\, d\rho^2 + \rho^2 d\Omega^2, \tag{19}$$

where

$$\begin{aligned} F_0(\rho) &= \frac{(2\rho - 3M)\, H_0(\rho) + M H_1(\rho)}{2\,(\rho - 2M)}, \\ F_1(\rho) &= \frac{(\rho - 3M)\, H_0(\rho) + (3\rho - 7M)\, H_1(\rho) + \rho(\rho - 2M)\,({H_0}'(\rho) + {H_1}'(\rho))}{2\,(\rho - 2M)}. \end{aligned}$$

Thus the metric perturbation in this gauge is

$$h^\mu{}_\nu = Diag\,[F_0(\rho), F_1(\rho), 0, 0]\,. \tag{20}$$

With this new gauge we see that to hold the surface area fixed when the metric perturbation is switched on is a trivial task: we simply hold ρ_b fixed.

In the introduction, we saw that physical negative modes are determined by the eigenvalue equation (4), which involves only the transverse traceless part of the perturbation. An infinitesimal coordinate transformation changes the metric by $g_{\mu\nu} \to g_{\mu\nu} + \epsilon \nabla_{(\mu}\xi_{\nu)}$ for some vector field ξ^μ, and thus only changes the longitudinal part. That is, although the metric perturbation in our new gauge choice looks more complicated, the equation we need to solve is unchanged, as the transverse traceless part is not affected by the gauge transformation. The transverse traceless part of (20) is just (13) with $r \to \rho$, that is,

$$h^{TT\,\mu}{}_\nu = Diag \left[H_0(\rho), H_1(\rho), -\frac{1}{2}\,(H_0(\rho) + H_1(\rho))\,, -\frac{1}{2}\,(H_0(\rho) + H_1(\rho)) \right] \tag{21}$$

with the transversality condition

$$H_0(\rho) = \left[-\frac{\rho(\rho - 2M)}{\rho - 3M}\frac{d}{d\rho} - \frac{3\rho - 5M}{\rho - 3M} \right] H_1(\rho). \tag{22}$$

We are thus seeking negative eigenvalues of the equation

$$\left[-\left(1-\frac{2M}{\rho}\right)\frac{d^2}{d\rho^2}-\frac{2(\rho-4M)(2\rho-3M)}{\rho^2(\rho-3M)}\frac{d}{d\rho}+\frac{8M}{\rho^2(\rho-3M)}\right]H_1(\rho)=\lambda H_1(\rho), \tag{23}$$

subject to the boundary conditions of regularity at $\rho = 2M$ and the isothermal boundary condition at the fixed radial position $\rho = \rho_b$. The first boundary condition is imposed by finding a regular series solution to the differential equation about $\rho = 2M$. There is only one such solution:

$$H_1(\rho)=\sum_{n=0}^{\infty}a_n(\rho/M-2)^n, \tag{24}$$

where

$$\begin{aligned}
a_1 &= -(\lambda M^2+2)a_0,\\
a_2 &= \frac{1}{6}(\lambda M^2+2)(2\lambda M^2+7)a_0,\\
a_3 &= -\frac{1}{36}(\lambda M^2+5)(2\lambda^2M^4+10\lambda M^2+15)a_0,\\
a_4 &= \frac{1}{360}(585+520\lambda M^2+168\lambda^2M^4+30\lambda^3M^6+2\lambda^4M^8)a_0.
\end{aligned}$$

The isothermal boundary condition fixes the proper length around the S^1 in the τ direction, which is given by $\sqrt{g_{\tau\tau}(\rho_b)}\Delta\tau$. It therefore imposes

$$\begin{aligned}
\left(1-\frac{2M}{\rho_b}\right) &= \left(1-\frac{2M}{\rho_b}\right)(1+F_0(\rho_b))\\
\Rightarrow F_0(\rho_b) &= 0\\
\Rightarrow H_0(\rho_b) &= \frac{M}{3M-2\rho_b}H_1(\rho_b).
\end{aligned} \tag{25}$$

H_0 is given in terms of H_1 by (22), so the isothermal boundary condition reduces to a mixed boundary condition for H_1,

$$\frac{{H_1}'(\rho_b)}{H_1(\rho_b)}+\frac{6{\rho_b}^2-20M\rho_b+18M^2}{\rho_b(\rho_b-2M)(2\rho_b-3M)}=0. \tag{26}$$

This can be contrasted to the condition derived by Allen by holding r_b fixed [**?**],

$$\frac{{H_1}'(r_b)}{H_1(r_b)}+\frac{3r_b-5M}{r_b(r_b-2M)}=0. \tag{27}$$

It should be noted that in deriving these conditions we have multiplied through by $\rho_b - 3M$ in (22) [$r_b - 3M$ in (14)]. This is acceptable so long as $\rho_b \neq 3M$, but for $\rho_b = 3M$, we need to consider more carefully the true boundary condition (25).

The method used in solving the differential equation is to numerically integrate from $\rho = 2M$ to $\rho = \rho_b$ using the form of the solution given by (24) as the initial data. However, the numerical method breaks down at $\rho = 3M$ due to the singularity here in the differential equation. To overcome this problem we find a power series solution about $\rho = 3M$ and find that the solution is in fact well behaved there. To evolve our solution through $\rho = 3M$ it is therefore necessary to numerically integrate up to $3M - \delta$ for some small δ and use the results of this numerical integration to fit the power series solution at $3M$ to the data. This power series can then be evaluated at $3M + \delta$ and the value of H_1 and its first derivative can be extracted there to provide new initial conditions for a numerical solution beginning at $3M + \delta$ which can then be evolved up to ρ_b.

Unlike at $\rho = 2M$ where only one of the series solutions was well behaved, there are two independent well behaved solutions at $\rho = 3M$. One of these is of order 1 whilst the other is $O\left((\rho - 3M)^3\right)$. Since the numerical solution breaks down in a fairly large region around $3M$, we can not approach the singularity with too small a δ. It is therefore necessary to go to fifth order in the series to maintain accuracy with a δ of order 10^{-3} and so the particular solution to the differential equation which we require here is a linear combination of both series. We therefore use the values of H_1 and ${H_1}'$ at $3M - \delta$ to find the coefficients b_0 and c_0 in the general form for the solution at $\rho = 3M$:

$$H_1(\rho) = \sum_{n=0}^{\infty} \left(b_n + c_n(\rho/M - 3)^3\right)(\rho/M - 3)^n, \tag{28}$$

where

$$b_1 = -\frac{4}{3}b_0, \qquad c_1 = -\frac{11}{6}c_0,$$

$$b_2 = \left(\frac{4}{3} + \frac{3}{2}\lambda M^2\right) b_0, \qquad c_2 = \left(\frac{7}{3} - \frac{3}{10}\lambda M^2\right) c_0,$$

$$b_3 = -\left(\frac{40}{27} + 5\lambda M^2\right) b_0,$$

$$b_4 = \left(\frac{130}{81} + \frac{15}{2}\lambda M^2 - \frac{9}{8}\lambda^2 M^4\right) b_0,$$

$$b_5 = -\left(\frac{136}{81} + \frac{79}{9}\lambda M^2 - 3\lambda^2 M^4\right) b_0.$$

The eigenvalue spectrum is now found by a shooting method. An arbitrary value of λ is input, the solution at $\rho = \rho_b$ is found and tested to see if it obeys the correct boundary condition. If it doesn't, λ is adjusted appropriately and the process is repeated until λ is found to the required precision. We have repeated this process with varying cavity sizes so that the value of the lowest eigenvalue can be plotted against ρ_b.

In figure 1, we give the results obtained for the boundary condition (27), demonstrating Allen's finding that the critical value of the radius is at $r_b \approx 2.89M$. In figure

2, we give the results for our boundary condition (26), showing clearly the expected result that there is a negative mode only for $\rho_b > 3M$.

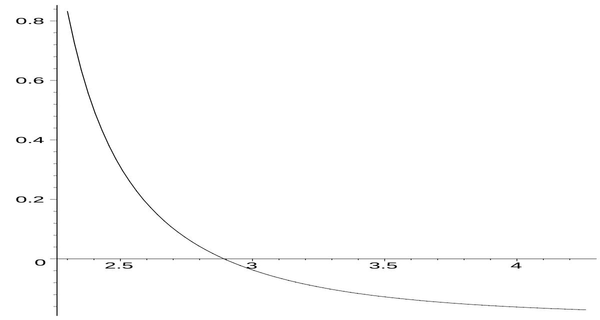

Figure 1: λM^2 vs ρ_b/M for the boundary condition (27)

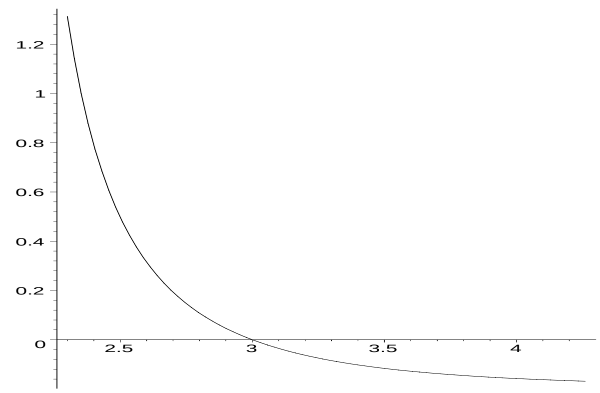

Figure 2: λM^2 vs ρ_b/M for the boundary condition (26)

3.2 Higher dimensions

We will now briefly discuss the extension of the results above to uncharged p-branes in $d+p$ dimensions for $d > 4$. We are interested in studying the negative modes of the Euclidean black hole geometry (9). In $d > 4$, the spherically symmetric transverse tracefree perturbation is

$$h^{\mu}{}_{\nu} = Diag\left[H_0(r), H_1(r), -\frac{1}{d-2}\left(H_0(r)+H_1(r)\right), \ldots, -\frac{1}{d-2}\left(H_0(r)+H_1(r)\right)\right], \tag{29}$$

and the transversality condition $\nabla_\mu h^\mu_\nu = 0$ gives

$$H_0(r) = -\frac{2r(r^{d-3}-\omega M)}{2r^{d-3}-(d-1)\omega M}H_1'(r) - \frac{2(d-1)r^{d-3}-(d+1)\omega M}{2r^{d-3}-(d-1)\omega M}H_1(r). \tag{30}$$

The Euclidean negative mode is still given by the eigenvectors of (4), which becomes

$$- \left(1 - \frac{\omega M}{r^{d-3}}\right) H_1''(r) - \frac{[2dr^{2d-6} - \omega M r^{d-3}(3d^2 - 11d + 18) + d(d-1)\omega^2 M^2]}{r^{d-2}(2r^{d-3} - (d-1)\omega M)} H_1'(r)$$
$$+ \frac{2d(d-3)^2 \omega M}{r^2(2r^{d-3} - (d-1)\omega M)} H_1(r) = \lambda H_1(r). \tag{31}$$

As in the four-dimensional case, it is convenient to make a coordinate transformation before applying the boundary conditions at the shell. Here the appropriate transformation is

$$r \to \rho : \quad \rho^2 = r^2 \left[1 - \frac{1}{d-2}\epsilon(H_0(r) + H_1(r))\right]. \tag{32}$$

The condition that the induced metric on the shell is fixed then implies that the shell is at some fixed $\rho = \rho_b$, and that

$$H_0(\rho_b) + \frac{d-3}{2(d-2)} \frac{\omega M}{\rho_b^{d-3} - \omega M}(H_0(\rho_b) + H_1(\rho_b)) = 0. \tag{33}$$

Using the transversality condition (30), this becomes a mixed boundary condition for H_1,

$$\frac{H_1'(\rho_b)}{H_1(\rho_b)} + \frac{2(d-1)(d-2)\rho_b^{2d-6} - 2(d^2-d-2)\omega M \rho_b^{d-3} + (d-1)^2\omega^2 M^2}{\rho_b(\rho_b^{d-3} - \omega M)[2(d-2)\rho_b^{d-3} - (d-1)\omega M]} = 0. \tag{34}$$

We analyse this eigenvalue problem using the same numerical methods as previously. The results are displayed in figure 3. In this graph the eigenvalue has been plotted against the cavity radius in a scaled and shifted radial coordinate, $\tilde{\rho}$, given by

$$\tilde{\rho} = \frac{\rho(\omega M)^{\frac{1}{3-d}} - 1}{\left(\frac{d-1}{2}\right)^{\frac{1}{d-3}} - 1}. \tag{35}$$

This choice is made so that in all dimensions we see that the black hole horizon is at $\tilde{\rho} = 0$ and the specific heat (11) is negative only when $\tilde{\rho}_b > 1$. It is clear that in all cases this corresponds exactly to the existence of the negative mode.

4 Conclusions

We have demonstrated that an uncharged black brane in a spherical cavity is classically unstable if and only if it is locally thermodynamically unstable. This provides a particularly simple and elegant example of the general connection between thermodynamic and classical instability conjectured in [**?**, **?**]. The key element of the proof was showing that as we vary the volume of the cavity, the classical instability disappears at precisely the point where the specific heat changes sign. To do this, we used the observation that the threshold unstable mode for the classical instability is the

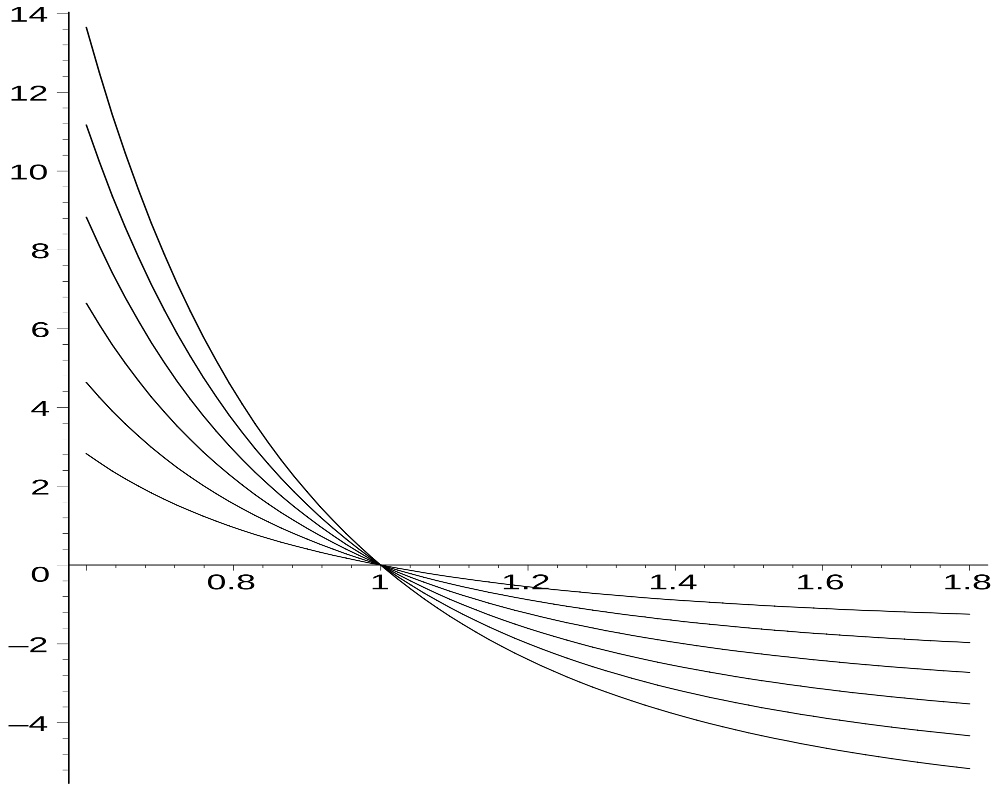

Figure 3: $\lambda(\omega M)^{\frac{2}{d-3}}$ vs $\tilde{\rho}_b$ for $d = 5$ (closest to axis) to $d = 10$ (furthest from axis) dimensions

analytic continuation of the Euclidean negative mode [?]. We then provided the first analysis of the Euclidean negative mode for Schwarzschild in a finite cavity using the boundary conditions appropriate to the canonical ensemble. (A similar calculation of the negative mode for the Euclidean Schwarzschild anti-de Sitter solution was carried out in [?]; unfortunately, it is not straightforward to construct a corresponding black brane solution in that case.)

One might be surprised that the presence of a boundary affects the classical instability at all; after all, the bulk metric is unchanged, so the initial behaviour of a perturbation with compact support near the horizon should be unaffected by the introduction of the boundary. However, in general relativity, the initial data are subject to constraints, so we are not free to specify an arbitrary initial perturbation of compact support, and the boundary conditions can influence the allowed possibilities for initial perturbations. It is hence reasonable that the boundary can turn off the classical instability.

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